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# ONE-DIMENSIONAL SHOCK WAVES IN UNIFORMLY DISTRIBUTED GRANULAR MATERIALS

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**Abstract**—Using a one-dimensional model for uniformly distributed granular solids, it is shown that the discontinuity in the porosity must be second order across a shock wave. This result implies that the Hugoniot properties of shock waves and their growth and decay behavior are qualitatively similar to those for homogeneous nonlinear elastic solids, but are quantitatively influenced by the initial porosity.

## 1. INTRODUCTION

In this study we continue our investigation of wave propagation in granular materials† and examine the behavior of one-dimensional shock waves in a granular material with an initially uniform distribution of granules. The basis for the continuum theory employed is the concept of a distributed body proposed by Goodman and Cowin[2] for flows of granular materials in which the porous nature of the material is represented in terms of the volume fraction of the granules. Here we consider an adaptation of this model appropriate for porous solids, pressed powders, and granular materials subject to sufficiently high confining pressure where the skeletal material is elastic.

In Section 2 we review the kinematics and field equations appropriate for this class of granular materials. We then state the constitutive assumption and note the restrictions imposed by thermodynamic considerations. In Section 3 the general properties of shock waves are considered including a discussion of the Hugoniot functions consistent with the concept of a distributed body. In particular we show that the discontinuity of the volume fraction at the shock must be second-order. As a result, the Hugoniot properties of shock waves in granular materials are similar to those for homogeneous nonlinear elastic solids. Finally, we consider the growth and decay of shock waves in Section 4 and show that the amplitude behavior is also similar to that of shock waves moving in elastic nonconductors. It is important to realize, however, that the material response functions depend on the initial porosity of the solid and this will affect the wave behavior indirectly. We also note that the elastic-type of behavior of the shock front appears to be consistent with experimental observations on porous polyurethane[3], porous aluminum[4], and on dry sand with sufficient confinement[5, 6].

## 2. CONSTITUTIVE ASSUMPTION AND FIELD EQUATIONS

Here we consider the one-dimensional response of a granular material with voids. We assign the material the structure of a distributed body[2] and, thus, represent the material as a continuum. The motion of the material can be described by the single field

$$x = \chi(X, t)$$

giving the spatial position $x$ at time $t$ of the particle which occupied the position $X$ in the reference configuration. For sufficiently smooth motions the particle velocity $v$ and the total strain $\epsilon$ are defined by

$$v = \dot{x} = \chi_t(X, t), \quad \epsilon = 1 - \chi_X(X, t). \tag{2.1}$$

†The growth and decay of one-dimensional acceleration waves is considered in [1].

It should be noted that in writing $(2.1)_2$ we have followed the common practice in shock wave physics and taken the strain $\epsilon(X, t)$ to be positive in compression. To be consistent with this, we shall also take the stress $\sigma(X, t)$ positive in compression.

An important consequence of the notion of a distributed body is that the bulk density $\rho$ at any point $X$ and time $t$ can be written as a product of the fields $\nu$ and $\gamma$:

$$\rho = \nu\gamma \tag{2.2}$$

where $\gamma = \gamma(X, t)$ is the *density of the granules* and $\nu = \nu(X, t)$, $0 < \nu \leq 1$, is the *volume fraction of the granules*.† The field $\nu(X, t)$ accounts for changes in the void volume such as compaction or distention. By the balance of mass and (2.2), the total strain is related to $\nu$ and $\gamma$ by

$$\frac{\rho_0}{\rho} = \frac{\nu_0\gamma_0}{\nu\gamma} = 1 - \epsilon \tag{2.3}$$

where $(\,)_0$ indicates evaluation in the reference configuration. It is important to note from (2.3) that both the total strain $\epsilon$ and the volume fraction $\nu$ must be specified in order to calculate the density of the granules $\gamma$. Thus, $\epsilon$ and $\nu$ are kinematically independent variables and this fact will necessitate the introduction of an additional force balance equation governing the void collapse. This equation involves a higher-order stress $h$ and a body force $g$ which must be specified as part of our constitutive assumption (see Goodman and Cowin [2]).

Throughout this study, we consider granular materials which do not conduct heat and assume that their response can be characterized by constitutive equations for the internal energy $e$, the stress $\sigma$, the absolute temperature $\theta$, the equilibrated stress $h$, and the intrinsic equilibrated body force $g$‡:

$$\begin{aligned} e &= \hat{e}(\nu_0, \nu, \nu_X, \dot{\nu}, \epsilon, \eta),\\ \sigma &= \hat{\sigma}(\nu_0, \nu, \nu_X, \dot{\nu}, \epsilon, \eta),\\ \theta &= \hat{\theta}(\nu_0, \nu, \nu_X, \dot{\nu}, \epsilon, \eta),\\ h &= \hat{h}(\nu_0, \nu, \nu_X, \dot{\nu}, \epsilon, \eta),\\ g &= \hat{g}(\nu_0, \nu, \nu_X, \dot{\nu}, \epsilon, \eta), \end{aligned} \tag{2.4}$$

where $\nu_0$ is the reference value of the volume fraction, $\nu$ is the present value of the volume fraction, $\nu_X$ is the volume fraction gradient, $\dot{\nu}$ is the volume fraction rate, $\epsilon$ is the strain and $\eta$ is the entropy. We further suppose that the response functions $\hat{e}$, $\hat{\sigma}$, $\hat{\theta}$, and $\hat{g}$ are even functions of the gradient $\nu_X$, i.e.§

$$\hat{A}(\cdot, \cdot, \nu_X, \cdot, \cdot, \cdot) = \hat{A}(\cdot, \cdot, -\nu_X, \cdot, \cdot, \cdot), \quad A = \epsilon, \sigma, \theta, g, \tag{2.5}$$

and that the response function $\hat{h}$ is an odd function of $\nu_X$, i.e.

$$\hat{h}(\cdot, \cdot, \nu_X, \cdot, \cdot, \cdot) = -\hat{h}(\cdot, \cdot, -\nu_X, \cdot, \cdot, \cdot). \tag{2.6}$$

The condition (2.6) implies that the equilibrated stress $h$ has the representation

$$h = 2\hat{\alpha}(\nu_0, \nu, \nu_X, \dot{\nu}, \epsilon, \eta)\nu_X \tag{2.7}$$

where $\alpha$ is an even function of $\nu_X$ and is assumed to be strictly positive:

$$\alpha > 0. \tag{2.8}$$

†The porosity $n$ of the material is related to the volume fraction $\nu$ by the formula, $n = 1 - \nu$.

‡These constitutive equations are slightly more general than those used in [1] to study acceleration waves in that we include here a dependence on $\dot{\nu}$.

§These conditions, in a three-dimensional context, imply that the material has a center of symmetry.

Hereafter the response function $\hat{e}$ is assumed to be of class $C^3$ and the fields $\chi$, $\nu$, and $\eta$ are assumed to be as smooth as the context requires.

An admissible thermodynamic process in a granular material is defined as any ordered array $\{\chi, \nu, \eta, e, \sigma, \theta, h, g, k\}$ which satisfies

$$\rho_0 \dot{v} = -\sigma_X + b, \tag{2.9}$$

$$\rho_0 k \ddot{\nu} = h_X + g + l, \quad \dot{k} = 0, \tag{2.10}$$

$$\dot{e} = \sigma \dot{\epsilon} + h \dot{\nu}_X - g \dot{\nu} + r, \tag{2.11}$$

$$\theta \dot{\eta} \geq r, \tag{2.12}$$

where $b$ and $r$ represent the extrinsic body force and heat supply, $l$ is the extrinsic equilibrated body force, and $k = k(X) > 0$ is the equilibrated inertia. It is well known that the entropy inequality (2.12) can only hold for every thermodynamic process if the response functions (2.4) satisfy certain restrictions. These restrictions are outlined in

*Theorem* 1. *Every thermodynamic process in a granular material is an admissible process if and only if*

(i) $\hat{e}$, $\hat{\sigma}$, $\hat{\theta}$ *and* $\hat{h}$ *are independent of* $\dot{\nu}$,

(ii)

$$\sigma = \hat{e}_\epsilon, \tag{2.13}$$

$$\theta = \hat{e}_\eta, \tag{2.14}$$

$$h = \hat{e}_{\nu_X}, \tag{2.15}$$

*and*

(iii)

$$(g + \hat{e}_\nu)\dot{\nu} \leq 0. \tag{2.16}$$

The proof of these results is fairly standard and will be omitted here (see Refs. [2] and [1]). Equations (2.13) and (2.14) are the usual stress and temperature relations which arise in studies of homogeneous materials, while (2.15) and (2.16) show the thermodynamic restrictions on how the internal energy $e$ must depend on the volume fraction $\nu$ and its gradient $\nu_X$.

In view of the smoothness of the internal energy function $\hat{e}$ and Theorem 1, we can define several important material properties. In particular, the *tangent modulus* $E$, the *second-order modulus* $\bar{E}$ and the *stress-entropy modulus* $G$ are given by

$$E = \hat{\sigma}_\epsilon, \quad \bar{E} = \hat{\sigma}_{\epsilon\epsilon}, \quad G = \hat{\sigma}_\eta, \tag{2.17}$$

and we assume that for all $(\nu_0, \nu, \nu_X, \epsilon, \eta)$

$$E > 0, \quad \bar{E} \neq 0, \quad G \neq 0. \tag{2.18}$$

Furthermore, we require that

$$E > 2\frac{\alpha}{k}. \tag{2.19}$$

There has been considerable discussion as to the physical interpretation of the equilibrated balance equation (2.10) and the quantities $k$, $h$, and $g$.† Clearly, this equation determines the compaction or distention of the voids and comparison with the recent work of Carroll and Holt[7] on porous materials suggests that $k$ is related to the initial surface area of the voids. It is also evident from (2.10) that the body force $g$ provides the coupling between the total deformation of the material and the compaction or distention of the void volume.‡ This force, then, would be associated with the actual forces acting in the skeletal material. In the case of granular media, these would be the Hertzian-type of contact forces acting on the granules. The fact that $g$ depends on $\dot{\nu}$ merely accounts for the frictional effects associated with these forces.

†In this regard, see Goodman and Cowin[2], and Passman[7].

‡The fact that this coupling must be defined in terms of a constitutive equation was first suggested by Herrmann[9].

However, if the void distribution is non-uniform, the granules are non-spherical, or the granules are of different sizes, there will be other contact forces acting on the surface of the granules which will tend to change the packing or the fabric of the material. Similar forces occur in porous materials which result in rupture of the cell structure. We associate the stress $h$ with these forces and it is clear that it is this quantity which controls the dilatancy of the material. This interpretation of the equilibrated stress $h$ is supported by the recent work of Cowin and Nunziato[10] who showed that the parameter $\alpha$ determines the speed of a wave of dilatancy in materials with incompressible granules (see also Nunziato and Walsh[1]). It is for this reason that we call $\alpha$ the *modulus of dilatancy*.† Then, the inequality (2.19) merely states the physical assumption that the sound speed associated with the compressibility of the granular material is greater than the speed of a wave of dilatancy associated with changes in the void volume.

## 3. GENERAL PROPERTIES OF SHOCK WAVES

The motion $\chi(X, t)$ is said to contain a *shock wave* if the function $\chi$ and the volume fraction $\nu$ are continuous for all $X$ and $t$, while, $v$, $\epsilon$, $\nu_X$, $\eta$ and their derivatives are continuous for all $X$ and $t$ except at points represented by the smooth one-parameter family $Y(t)$ where they may suffer jump discontinuities. The function $Y(t)$ gives the location in the reference configuration at which the wave is to be found at time $t$ and hence, represents the material trajectory of the wave. The wave propagates with the intrinsic velocity

$$U = U(t) = \frac{\mathrm{d}Y(t)}{\mathrm{d}t} > 0. \tag{3.1}$$

We denote the jump discontinuity in a function $f(X, t)$ by

$$[f] = f^- - f^+, \quad f^\pm = \lim_{X \to Y^\pm(t)} f(X, t),$$

and, if $\dot{f}$ and $f_X$ are also discontinuous at the wave, then

$$\frac{\mathrm{d}}{\mathrm{d}t}[f] = [\dot{f}] + U[f_X]. \tag{3.2}$$

With $U > 0$, $f^+$ and $f^-$ are the limiting values of $f(X, t)$ immediately ahead of and behind the wave. We note that at a shock wave the functions $\hat{e}$, $\hat{\sigma}$, $\hat{\theta}$, $\hat{g}$ and $\hat{h}$ may also be discontinuous.

Assuming that all external forces and supplies are absent, the balance relations (2.9)–(2.11) imply that across a shock[1]

$$\rho_0 U[v] = [\sigma] \tag{3.3}$$

$$\rho_0 U[k\dot{\nu}] = -[h] \tag{3.4}$$

$$U[k] = 0, \tag{3.5}$$

$$U\left[e + \frac{\rho_0}{2}v^2 + \frac{\rho_0}{2}k\dot{\nu}^2\right] = [\sigma v] - [h\dot{\nu}] \tag{3.6}$$

$$[\eta] \geq 0, \tag{3.7}$$

and

$$\rho_0[\dot{v}] = -[\sigma_X], \tag{3.8}$$

$$\rho_0[k\ddot{\nu}] = [h_X] + [g], \tag{3.9}$$

$$[\dot{k}] = 0, \tag{3.10}$$

$$[\theta\dot{\eta}] = -[(g + \hat{e}_\nu)\dot{\nu}]. \tag{3.11}$$

†In a recent study of the flows of granular materials, Cowin[11] introduced a quantity $\zeta$ which he called the *dilatancy modulus*. This modulus is related to our modulus of dilatancy $\alpha$ by $\zeta = \alpha_0$, $\alpha_0$ being the value of $\alpha$ in the reference configuration. configuration.

Since $\chi$ and $\nu$ are continuous, (2.1) and (3.2) assert that the kinematical relations

$$[v] = U[\epsilon], \quad [\dot{\nu}] = -U[\nu_X] \tag{3.12}$$

hold at the wave. Combining these expressions with (3.3)–(3.6) we obtain the relations

$$\rho_0 U^2 = \frac{[\sigma]}{[\epsilon]}, \tag{3.13}$$

$$[h] = k\frac{[\sigma]}{[\epsilon]}[\nu_X], \tag{3.14}$$

$$[e] = \frac{1}{2}(\sigma^- + \sigma^+)[\epsilon] + \frac{1}{2}(h^- + h^+)[\nu_X], \tag{3.15}$$

where the equilibrated inertia $k$ is continuous. These results are to be looked upon as Hugoniot relations in that by using the appropriate constitutive equations $(2.4)_{1,2}$ and (2.7), we can determine the velocity $U$ and the jumps in volume fraction gradient $\nu_X$ and entropy $\eta$ in terms of the jump in strain $\epsilon$. Clearly, $[\epsilon]$ is a measure of the wave amplitude.

Throughout this study, we will confine our attention to the propagation of a shock wave into material which is in its reference configuration and is undeformed and stress-free. Then

$$\epsilon^+ = 0, \quad \eta^+ = \eta_0, \quad \nu_X{}^+ = 0, \quad \nu^+ = \nu_0,$$

and

$$\begin{aligned} e^- &= \hat{e}(\nu_0, \nu_0, \nu_X{}^-, \epsilon^-, \eta^-), & e^+ &= e_0, \\ \sigma^- &= \hat{\sigma}(\nu_0, \nu_0, \nu_X{}^-, \epsilon^-, \eta^-), & \sigma^+ &= 0, \\ h^- &= 2\hat{\alpha}(\nu_0, \nu_0, \nu_X{}^-, \epsilon^-, \eta^-)\nu_X{}^-, & h^+ &= 0. \end{aligned} \tag{3.16}$$

If we now define the Hugoniot functions

$$\hat{H}(\epsilon, \eta, \nu_X) = \hat{e}(\nu_0, \nu_0, \nu_X, \epsilon, \eta) - e_0 - \frac{1}{2}\hat{\sigma}(\nu_0, \nu_0, \nu_X, \epsilon, \eta)\epsilon - \hat{\alpha}(\nu_0, \nu_0, \nu_X, \epsilon, \eta)\nu_X{}^2, \tag{3.17}$$

$$\hat{J}(\epsilon, \eta, \nu_X) = \left\{2\hat{\alpha}(\nu_0, \nu_0, \nu_X, \epsilon, \eta) - k\frac{\hat{\sigma}(\nu_0, \nu_0, \nu_X, \epsilon, \eta)}{\epsilon}\right\}\nu_X, \tag{3.18}$$

then, by (3.14)–(3.16), the intersection of the surfaces

$$\hat{H}(\epsilon^-, \eta^-, \nu_X{}^-) = 0, \tag{3.19}$$

$$\hat{J}(\epsilon^-, \eta^-, \nu_X{}^-) = 0, \tag{3.20}$$

represent all states $(\epsilon^-, \eta^-, \nu_X{}^-)$ attainable in a shock jump from the initial state $(0, \eta_0, 0)$. It should be noted that by virtue of the smoothness of the response function $\hat{\epsilon}$, the Hugoniot functions $\hat{H}(\epsilon, \eta, \nu_X)$ and $\hat{J}(\epsilon, \eta, \nu_X)$ are single-valued, continuous, and differentiable functions of their arguments. We are now in a position to prove our main result.

*Theorem* 2. *Consider a shock wave propagating into an uniformly distributed granular material in its reference configuration. Then if* (2.19) *holds, the discontinuity in the volume fraction $\nu$ is at least second-order; that is*

$$[\nu_X] = \nu_X{}^- \equiv 0. \tag{3.21}$$

*Proof.* In view of the smoothness of the functions $\hat{H}(\epsilon, \eta, \nu_X)$ and $\hat{J}(\epsilon, \eta, \nu_X)$, it is clear from (3.18) and (3.20) that it suffices to show that

$$I(\epsilon^-, \eta^-, \nu_X{}^-) = 2\hat{\alpha}(\nu_0, \nu_0, \nu_X{}^-, \epsilon^-, \eta^-) - k\frac{\hat{\sigma}(\nu_0, \nu_0, \nu_X{}^-, \epsilon^-, \eta^-)}{\epsilon^-} \neq 0 \tag{3.22}$$

for any state $(\epsilon^-, \eta^-, \nu_X^-)$. Therefore, let us consider the behavior of the shock in the neighborhood of the initial state $(0, \eta_0, 0)$. Since both the surfaces $\hat{H}(\cdot,\cdot,\cdot)$ and $\hat{J}(\cdot,\cdot,\cdot)$ pass through this state, the implicit function theorem implies the existence of the functions

$$\eta^- = N_H(\epsilon^-), \quad \nu_X^- = M_H(\epsilon^-) \tag{3.23}$$

provided at $(0, \eta_0, 0)$,

$$\begin{vmatrix} \hat{H}_\eta & \hat{H}_{\nu_X} \\ \hat{J}_\eta & \hat{J}_{\nu_X} \end{vmatrix} \neq 0. \tag{3.24}$$

That this latter condition does hold is evident from the fact that (3.17), (3.18), (2.14) and (2.20) yield

$$\begin{vmatrix} \hat{H}_\eta & \hat{H}_{\nu_X} \\ \hat{J}_\eta & \hat{J}_{\nu_X} \end{vmatrix} = \theta_0(2\alpha_0 - kE_0) < 0$$

at $(0, \eta_0, 0)$. Furthermore, we note that the functions $N_H(\cdot)$ and $M_H(\cdot)$ have bounded derivatives. tives.

Now, by virtue of (3.16), (3.23), (2.13), (2.15), (2.7) and (2.17),

$$[\eta] = N_H'(0)\epsilon^- + \cdots, \tag{3.25}$$

$$\begin{aligned}\sigma^- &= E_0\epsilon^- + G_0[\eta] + \cdots,\\ &= \{E_0 + G_0 N_H'(0)\}\epsilon^- + \cdots.\end{aligned} \tag{3.26}$$

By differentiating (3.19), we also have

$$\begin{aligned}2\theta N_H'(\epsilon^-) = -\sigma^- &+ \{E^- + G^- N_H'(\epsilon^-) + (\hat{\sigma}_{\nu_X})^- M_H'(\epsilon^-)\}\epsilon^-\\ &+ 2\{(\hat{\alpha}_\epsilon)^- \nu_X^- + (\hat{\alpha}_\eta)^- N_H'(\epsilon^-)\nu_X^-\\ &+ ((\hat{\alpha}_{\nu_X})^- \nu_X^- + 2\alpha^-) M_H'(\epsilon^-)\}\nu_X^-\end{aligned} \tag{3.27}$$

and, thus, as $\epsilon^- \to 0$, $\eta^- \to \eta_0$, $\nu_X^- \to 0$, (3.26) and (3.27) assert that

$$N_H'(0) = 0. \tag{3.28}$$

Using (3.22), (3.26), (3.28) and (2.19), it is clear then that

$$I(0, \eta_0, 0) = 2\alpha_0 - kE_0 < 0$$

and hence the proof is complete.

The continuity of $\nu_X$ across a shock of arbitrary amplitude is an extremely important result; for it implies that (3.20) is trivially satisfied and (3.19) reduces to the familiar Hugoniot relation

$$\begin{aligned}\bar{H}(\epsilon^-, \eta^-) = \hat{e}(\nu_0, \nu_0, 0, \epsilon^-, \eta^-) &- e_0\\ &- \frac{1}{2}\hat{\sigma}(\nu_0, \nu_0, 0, \epsilon^-, \eta^-)\epsilon^-.\end{aligned} \tag{3.29}$$

Since $G \neq 0$, $(2.18)_3$ further implies the existence of the inverse function

$$\eta = \hat{\sigma}^{-1}(\nu_0, \nu, \nu_X, \epsilon, \sigma). \tag{3.30}$$

and thus we can define the *Hugoniot stress–strain curve*:

$$\tilde{H}(\epsilon^-, \sigma^-) = \bar{H}(\epsilon^-, \hat{\sigma}^{-1}(\nu_0, \nu_0, 0, \epsilon^-, \sigma^-)) = 0. \tag{3.31}$$

Clearly, from (3.13) the shock velocity $U$ is related to the secant (Rayleigh line) connecting the initial state $(0, 0)$ and the final state $(\epsilon^-, \sigma^-)$ on the Hugoniot curve. Nunziato and Herrmann[12] have studied the general properties of the Hugoniot curve defined by (3.31) and have proved the contents of

*Theorem* 3. *Consider a shock wave propagating into undeformed material at rest which is also stress-free and suppose that* (2.8), (2.18), *and* (2.19) *hold. Then*

(i) *if* $\hat{E} > 0$ *for all* $\epsilon$, *the shock is compressive, i.e.*

$$\epsilon^- > 0;$$

*on the other hand, if* $\hat{E} < 0$ *for all* $\epsilon$, *the shock is expansive, i.e.*

$$\epsilon^- < 0;$$

(ii) *the shock velocity and the entropy increase monotonically as the Hugoniot stress-strain curve is traversed outward from the origin* $(0, 0)$; *and*

(iii) *the shock velocity is supersonic with respect to the material ahead of the wave and subsonic with respect to the material behind the wave, i.e.*

$$E_0 < \rho_0 U^2 < E^-.$$

Having established these results, we can now consider the growth and decay of shocks of arbitrary amplitude $\epsilon^-$.

## 4. THE GROWTH AND DECAY OF SHOCK WAVES

In this section we derive a differential equation which governs the amplitude of the shock wave propagating in a uniformly distributed granular material. The derivation of this equation is fairly standard (see for example, Chen and Gurtin[13] and so will only be outlined here. Throughout the analysis it will be assumed that the material ahead of the wave is in its reference configuration which corresponds to a state of zero strain, zero stress and constant entropy.

Using (3.2) with $f = \epsilon$ and $v$ and combining this with (3.8), the following relation for the amplitude $\epsilon^-$ of the shock is obtained:

$$2U\frac{d\epsilon^-}{dt} + \epsilon^-\frac{dU}{dt} = \left(U^2 - \frac{E^-}{\rho_0}\right)[\epsilon_X] - \frac{G^-}{\rho_0}[\eta_X] - \frac{(\hat{\sigma}_{\nu_X})^-}{\rho_0}[\nu_{XX}] \tag{4.1}$$

where we have used (2.18). At this point we note that, by (2.7) and the thermodynamic results (2.13) and (2.15),

$$(\hat{\sigma}_{\nu_X})^- = (\hat{h}_\epsilon)^- = 2(\hat{\alpha}_\epsilon)^-(\nu_X)^-.$$

However, in view of Theorem 2, the volume fraction gradient $\nu_X$ must be continuous across the shock and thus it follows that $(\hat{\sigma}_{\nu_X})^- = 0$ for a wave entering undeformed material. In this case (4.1) reduces to the form given by Chen and Gurtin (eqn (3.16) of [13]) for a shock entering an elastic nonconductor. The subsequent amplitude relation is obtained from (4.1) by evaluating $dU/dt$ and $[\eta_X]$. In particular, using (3.11), (3.2) with $f = \eta$ and $e$, (3.15), (3.16) and Theorem 2, it can be shown that

$$[\eta_X] = \frac{1}{U}\frac{d[\eta]}{dt} = \frac{E^-(\mu - 1)}{G^-(1 - 2\tau)U}\frac{d\epsilon^-}{dt}, \tag{4.2}$$

and

$$\frac{dU}{dt} = \frac{U(\mu - 1)\tau}{\mu\epsilon^-(1 - 2\tau)}\frac{d\epsilon^-}{dt}, \tag{4.3}$$

where

$$\tau = \frac{\theta^-}{G^- \epsilon^-}, \quad \mu = \frac{\rho_0 U^2}{E^-}. \tag{4.4}$$

Substituting (4.2)–(4.4) into (4.1) gives the amplitude equation

$$\frac{d\epsilon^-}{dt} = -\frac{U(1-\mu)(2\tau-1)}{\tau(3\mu+1)-(3\mu-1)}[\epsilon_X]. \qquad (4.5)\dagger$$

As expected, the amplitude behavior is determined by the strain gradient behind the wave; what is somewhat surprising is that the amplitude does not depend explicitly on the granularity and thus the behavior of the shock front is qualitatively the same as it would be in a homogeneous elastic material. Of course, there is an influence on the amplitude due to the initial porosity as it affects the material response functions $E$ and $G$, and the shock velocity $U$.

To illustrate the growth and decay behavior of the wave, as predicted by (4.5), we note from result (iii) of Theorem 3 that

$$\mu = \frac{\rho_0 U^2}{E^-} < 1. \tag{4.6}$$

Further, we assume, as is true for most materials, that $G^- > 0$ and that the Hugoniot relation (3.31) is single-valued. Then, by (4.2), (2.18) and (ii) of Theorem 3, it follows that for compressive shocks [12]

$$\tau > \frac{1}{2}. \tag{4.7}$$

Equations (4.5)–(4.7) combine to yield

*Theorem 4. Consider a compressive shock wave propagating into an undeformed granular material at rest which is also stress-free. Then, if $G^- > 0$ and (2.19) holds*

(i) $$[\epsilon_X] > 0 \Leftrightarrow \frac{d\epsilon^-}{dt} < 0,$$

(ii) $$[\epsilon_X] < 0 \Leftrightarrow \frac{d\epsilon^-}{dt} > 0.$$

(iii) $$[\epsilon_X] = 0 \Leftrightarrow \frac{d\epsilon^-}{dt} = 0.$$

Figure 1 is a schematic of the situation described in Theorem 4. Note in particular that,

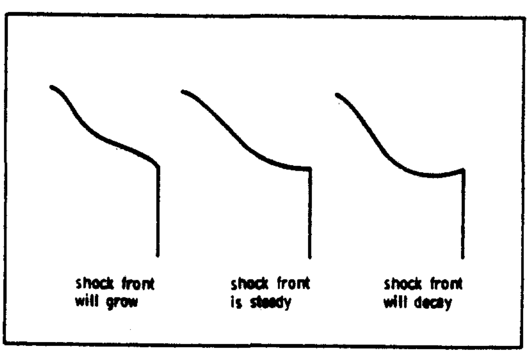


Fig. 1. Shock wave profiles $\epsilon(X)$ in a uniformly distributed granular material with $\hat{E}^- > 0$, $G^- > 0$ at a specific time $t$. The wave is propagating into unstrained material at rest and the slope immediately behind the shock jump is $[\epsilon_X]$.

†Chen and Gurtin[13], eqn (3.16).

although the shock front propagates as an elastic wave, we expect the influence of the porosity to be felt behind the front in a manner shown qualitatively in Fig. 1. In a recent numerical study, Nunziato and Yarrington[13] used the present theory to model porous materials to show this type of two-wave structure and compared the results with experimental observations on polyurethane foam[3]. Similar wave structures have also been observed in porous aluminum[4] and in dry sand under certain conditions[5, 6].† These observations appear to be consistent with the result that the discontinuity in the volume fraction, resulting from the pore collapse initiated by the shock wave, must be second-order. We can calculate this discontinuity from (3.9), using (3.24), (3.12) and (2.7),

$$[\ddot{\nu}] = \frac{U^2}{(\rho_0 k U^2 - 2\alpha^-)}[g].$$

This result further emphasizes the role of the intrinsic equilibrated body force $g$ in the pore collapse process associated with a propagating compressive wave.

Finally, we wish to note that in the case of weak shock waves all the results obtained for elastic non-conductors apply to the present theory. That is, for small amplitudes $\epsilon^-$, the shock velocity $U$ is given by

$$U = \sqrt{\left(\frac{E_0}{\rho_0}\right)} + O(\epsilon^-);$$

the entropy jump $[\eta]$ is third-order in the amplitude $\epsilon^-$,

$$[\eta] = \frac{\bar{E}_0}{12\theta_0}(\epsilon^-)^3 + O((\epsilon^-)^4);$$

and the shock amplitude $\epsilon^-$ is constant. These results are consistent with the exact solution to the impact problem obtained by Nunziato and Walsh[15] in the context of the linearized theory of granular materials.

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†It is important to note that the results given here apply only to materials in which $\bar{E} > 0$ for all $\epsilon$ of interest. However, in some granular materials (see Seaman and Whitman[5]), the curvature of the stress–strain curve is not positive everywhere and thus the shock wave behavior in these materials can be much more complex.